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Weak convergence of algorithms for asymptotically strict pseudocontractions in the intermediate sense and equilibrium problems

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Abstract

In this paper, equilibrium problems and fixed-point problems based on an iterative method are investigated. It is proved that the sequence generated in the purposed iterative process weakly converges to a common element of the fixed-point set of an asymptotically strict pseudocontraction in the intermediate sense and the solution set of a system of equilibrium problems in the framework of real Hilbert spaces.

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1 Introduction

Approximating solutions of nonlinear operator equations based on iterative methods is now a hot topic of intensive research efforts. Indeed, many well-known problems can be studied by using algorithms which are iterative in their nature. As an example, in computer tomography with limited data, each piece of information implies the existence of a convex set C_m in which the required solution lies. The problem of finding a point in the intersection $\bigcap_{m=1}^{N} C_m$, where $N \geq 1$ is some positive integer is of crucial interest, and it cannot be usually solved directly. Therefore, an iterative algorithm must be used to approximate such point. The well-known convex feasibility problem which captures applications in various disciplines such as image restoration, and radiation therapy treatment planning is to find a point in the intersection of common fixed-point sets of a family of nonlinear mappings (see [1–7]). There many classic algorithms, for example, the Picard iterative algorithm, the Mann iterative algorithm, the Ishikawa iterative algorithm, steepest descent iterative algorithms, hybrid projection algorithms, and so on. In this paper, we shall investigate fixed-point and equilibrium problems based on a Mann-like iterative algorithm.

The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, equilibrium problems and fixed-point problems of asymptotically strict pseudocontractions in the intermediate sense are discussed based on a Mann iterative algorithm. Weak convergence theorems are established in Hilbert spaces. And some deduced results are also obtained.

2 Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H and F a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ stands for the set of real numbers. In this paper, we consider the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{2.1}$$

The set of such an $x \in C$ is denoted by $\mathrm{EP}(F)$, *i.e.*,

$$\mathrm{EP}(F) = \{x \in C : F(x,y) \geq 0, \forall y \in C\}.$$

Given a mapping $A : C \to H$, let $F(x,y) = \langle Ax, y-x\rangle$ for all $x,y \in C$. Then $z \in \mathrm{EP}(F)$ if and only if

$$\langle Az, y-z\rangle \geq 0, \quad \forall y \in C,$$

that is, z is a solution of the classical variational inequality. In this paper, we use $\mathrm{VI}(C,A)$ to stand for the set of solutions of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem (2.1).

To study the equilibrium problem (2.1), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;
(A3) for each $x,y,z \in C$,

$$\limsup_{t\downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Let $S : C \to C$ be a mapping. In this paper, we use $F(S)$ to denote the fixed-point set of S. Recall the following definitions.

S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x-y\|, \quad \forall x,y \in C.$$

If C is a bounded, closed, and convex subset of H, then $F(S)$ is nonempty, closed, and convex. Recently, many beautiful convergence theorems for fixed points of nonexpansive mappings (semigroups) have been established in Banach spaces (see [8–10] and the references therein).

S is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - S^n y\| \leq k_n \|x-y\|, \quad \forall x,y \in C, n \geq 1.$$

It is known that if C is a nonempty, bounded, and closed convex subset of a Hilbert space H, then every asymptotically nonexpansive self-mapping has a fixed point. Further, the set

$F(S)$ of fixed points of S is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence problems of iterative processes for such a class of mappings.

S is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| S^n x - S^n y \right\| - \|x-y\| \right) \le 0. \tag{2.2}$$

Putting

$$\xi_n = \max\left\{ 0, \sup_{x,y\in C} \left(\left\| S^n x - S^n y \right\| - \|x-y\| \right) \right\}, \tag{2.3}$$

we see that $\xi_n \to 0$ as $n \to \infty$. Then (2.2) is reduced to the following:

$$\left\| S^n x - S^n y \right\| \le \|x-y\| + \xi_n, \quad \forall x,y \in C.$$

The class of asymptotically nonexpansive mappings in the intermediate sense was considered in [11] and [12] as a generalization of the class of asymptotically nonexpansive mappings. It is known that if C is a nonempty, closed, and convex bounded subset of a real Hilbert space, then every asymptotically nonexpansive self-mapping in the intermediate sense has a fixed point.

S is said to be strictly pseudocontractive if there exists a constant $\kappa \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x-y\|^2 + \kappa \left\| (I-S)x - (I-S)y \right\|^2, \quad \forall x,y \in C.$$

For such a case, S is also said to be a κ-strict pseudocontraction. It is clear that every nonexpansive mapping is a 0-strict pseudocontraction. We also remark that if $\kappa = 1$, then S is said to be pseudocontractive.

S is said to be an asymptotically strict pseudocontraction if there exist a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0,1)$ such that

$$\left\| S^n x - S^n y \right\|^2 \le k_n \|x-y\|^2 + \kappa \left\| \left(I - S^n\right)x - \left(I - S^n\right)y \right\|^2, \quad \forall x,y \in C, n \ge 1.$$

For such a case, S is also said to be an asymptotically κ-strict pseudocontraction. It is clear that every asymptotically nonexpansive mapping is an asymptotically 0-strict pseudocontraction. We also remark here that if $\kappa = 1$, then S is said to be an asymptotically pseudocontractive mapping which was introduced by Schu [13] in 1991.

S is said to be an asymptotically strict pseudocontraction in the intermediate sense if there exist a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0,1)$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| S^n x - S^n y \right\|^2 - k_n \|x-y\|^2 - \kappa \left\| \left(I - S^n\right)x - \left(I - S^n\right)y \right\|^2 \right) \le 0. \tag{2.4}$$

For such a case, S is also said to be an asymptotically κ-strict pseudocontraction in the intermediate sense. Putting

$$\xi_n = \max\left\{ 0, \sup_{x,y\in C} \left(\left\| S^n x - S^n y \right\|^2 - k_n \|x-y\|^2 - \kappa \left\| \left(I - S^n\right)x - \left(I - S^n\right)y \right\|^2 \right) \right\}, \tag{2.5}$$

then we see that $\xi_n \to 0$ as $n \to \infty$. We know that (2.4) is reduced to the following:

$$\|S^n x - S^n y\|^2 \le k_n \|x - y\|^2 + \kappa \|(I - S^n)x - (I - S^n)y\|^2 + \xi_n, \quad \forall x, y \in C, n \ge 1.$$

The class of asymptotically strict pseudocontractions in the intermediate sense was introduced by Sahu, Xu, and Yao [14] as a generalization of the class of asymptotically strict pseudocontractions; see [14] for more details.

Recently, many authors considered the weak convergence of iterative sequences for the classical variational inequality, the equilibrium problem (2.1), and fixed-point problems based on iterative methods (see [15–24]).

In 2003, Takahashi and Toyoda [23] considered the classical variational inequality and a nonexpansive mapping. To be more precise, they proved the following theorem.

Theorem 1 *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \quad \forall n \ge 0,$$

for every $n \ge 0$, where $\lambda_n \in [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\alpha_n \in [c, d]$ for some $c, d \in (0, 1)$. Then $\{x_n\}$ converges weakly to $z \in F(S) \cap \mathrm{VI}(C, A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap \mathrm{VI}(C,A)} x_n$.

In 2007, Tada and Takahashi [24] considered the equilibrium problem (2.1) and a nonexpansive mapping. To be more precise, they proved the following result.

Theorem 2 *Let C be a nonempty, closed, and convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let S be a nonexpansive mapping of C into H, such that $F(S) \cap \mathrm{EP}(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and let*

$$\begin{cases} u_n \in C \text{ such that } F(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S u_n, \end{cases}$$

for each $n \ge 1$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges weakly to $w \in F(S) \cap \mathrm{EP}(F)$, where $w = \lim_{n\to\infty} P_{F(S)\cap \mathrm{EP}(F)} x_n$.

In this paper, motivated by the results announced in [23] and [24], we consider the equilibrium problem (2.1) and an asymptotically strict pseudocontraction in the intermediate sense based on a Mann-like iterative process. We show that the sequence generated in the purposed iterative process converges weakly to a common element of the fixed-point set of an asymptotically strict pseudocontraction in the intermediate sense and the solution set of the equilibrium problem (2.1). The results presented in this paper improve and extend the corresponding results announced by Takahashi and Toyoda [23], and Tada and Takahashi [24].

In order to prove our main results, we also need the following lemmas.

The following lemma can be found in [25] and [26].

Lemma 2.1 *Let C be a nonempty, closed, and convex subset of H, and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following statements hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) *$F(T_r) = \mathrm{EP}(F)$;*

(d) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 2.2 ([14]) *Let H be a real Hilbert space, C a nonempty, closed, and convex subset of H, and $S : C \to C$ a uniformly continuous and asymptotically strict pseudo-contraction in the intermediate sense. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $x = Tx$.*

Lemma 2.3 ([27]) *Let H be a real Hilbert space, and $0 < p \leq t_n \leq q < 1$, for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \qquad \limsup_{n\to\infty} \|y_n\| \leq r$$

and for some $r \geq 0$,

$$\lim_{n\to\infty} \left\| t_n x_n + (1 - t_n) y_n \right\| = r.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.4 ([28]) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 2.5 ([29]) *Let H be a real Hilbert space. Let $\{a_n\}_{n=1}^N$ be real numbers in $[0,1]$ such that $\sum_{n=1}^N a_n = 1$. Then we have the following:*

$$\left\|\sum_{i=1}^N a_i x_i\right\|^2 \le \sum_{i=1}^N a_i\|x_i\|^2,$$

for any given bounded sequence $\{x_n\}_{n=1}^N$ in H.

3 Main results

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4) for each $1 \le m \le N$, where $N \ge 1$ is some positive integer. Let $S : C \to C$ be a uniformly continuous and asymptotically κ-strict pseudocontraction in the intermediate sense. Assume that $\mathcal{F} := F(S) \cap \bigcap_{m=1}^N EP(F_m)$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{\lambda_n\}$ be sequences in $[0,1]$, and $\{e_n\}$ a bounded sequence in C. Let $\{r_{n,m}\}$ be a positive sequence such that $\liminf_{n\to\infty} r_{n,m} > 0$ and $\{\gamma_{n,m}\}$ a sequence in $[0,1]$ for each $1 \le m \le N$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in H, \\ u_{n,m} \in C \text{ such that } F_m(u_{n,m}, u_m) + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \ge 0, \quad \forall u_m \in C, \\ z_n = \sum_{m=1}^N \gamma_{n,m} u_{n,m}, \\ x_{n+1} = \alpha_n x_n + \beta_n(\delta_n z_n + (1-\delta_n)S^n z_n) + \lambda_n e_n, \quad n \ge 1. \end{cases}$$

Assume that the following restrictions are satisfied:

(a) $\alpha_n + \beta_n + \lambda_n = 1$;
(b) $\sum_{n=1}^\infty (k_n - 1) < \infty$ *and* $\sum_{n=1} \xi_n < \infty$, *where ξ_n is defined in (2.5);*
(c) $0 < a \le \beta_n \le b < 1$, $0 \le \kappa \le \delta_n \le c < 1$ *and* $\sum_{n=1}^\infty \lambda_n < \infty$;
(d) $\sum_{m=1}^N \gamma_{n,m} = 1$ *and* $0 < d \le \gamma_{n,m} \le 1$ *for each* $1 \le m \le N$,

where a, b, c, and d are some real constants. Then the sequence $\{x_n\}$ weakly converges to some point in $\mathcal{F}$.

Proof Let $p \in \mathcal{F}$. Then we see that

$$\|z_n - p\| \le \sum_{m=1}^N \gamma_{n,m}\|u_{n,m} - p\| \le \|x_n - p\|. \tag{3.1}$$

Put $y_n = \delta_n z_n + (1-\delta_n)S^n z_n$ for each $n \ge 1$. In view of Lemma 2.5, we see from the restriction (c) that

$$\begin{aligned} \|y_n - p\|^2 &= \left\|\delta_n z_n + (1-\delta_n)S^n z_n - p\right\|^2 \\ &= \delta_n\|z_n - p\|^2 + (1-\delta_n)\left\|S^n z_n - p\right\|^2 - \delta_n(1-\delta_n)\left\|z_n - S^n z_n\right\|^2 \\ &\le \delta_n\|z_n - p\|^2 + (1-\delta_n)k_n\|z_n - p\|^2 + (\kappa - \delta_n)\left\|z_n - S^n z_n\right\|^2 + \xi_n \\ &\le k_n\|z_n - p\|^2 + \xi_n \\ &\le k_n\|x_n - p\|^2 + \xi_n. \end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+\beta_n\|y_n-p\|^2+\lambda_n\|e_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+\beta_n\big(k_n\|x_n-p\|^2+\xi_n\big)+\lambda_n\|e_n-p\|^2\\ &\le k_n\|x_n-p\|^2+\xi_n+\lambda_n\|e_n-p\|^2. \end{aligned}\tag{3.3}$$

From Lemma 2.4, we obtain the existence of the limit of the sequence $\{\|x_n-p\|\}$. Notice that

$$\begin{aligned}\|u_{n,m}-p\|^2 &= \|T_{r_{n,m}}x_n-T_{r_{n,m}}p\|^2\\ &\le \langle T_{r_{n,m}}x_n-T_{r_{n,m}}p, x_n-p\rangle\\ &= \langle u_{n,m}-p, x_n-p\rangle\\ &= \frac{1}{2}\big(\|u_{n,m}-p\|^2+\|x_n-p\|^2-\|u_{n,m}-x_n\|^2\big), \quad \forall 1\le m\le N.\end{aligned}$$

This implies that

$$\|u_{n,m}-p\|^2 \le \|x_n-p\|^2-\|u_{n,m}-x_n\|^2, \quad \forall 1\le m\le N. \tag{3.4}$$

In view of (3.4) and $z_n=\sum_{m=1}^N \gamma_{n,m}u_{n,m}$, where $\sum_{m=1}^N \gamma_{n,m}=1$, we see from Lemma 2.5 that

$$\begin{aligned}\|z_n-p\|^2 &\le \sum_{m=1}^N \gamma_{n,m}\|u_{n,m}-p\|^2\\ &\le \sum_{m=1}^N \gamma_{n,m}\big(\|x_n-p\|^2-\|u_{n,m}-x_n\|^2\big)\\ &\le \sum_{m=1}^N \gamma_{n,m}\|x_n-p\|^2-\sum_{m=1}^N \gamma_{n,m}\|u_{n,m}-x_n\|^2\\ &= \|x_n-p\|^2-\sum_{m=1}^N \gamma_{n,m}\|u_{n,m}-x_n\|^2.\end{aligned}\tag{3.5}$$

In view of Lemma 2.5, we obtain from restriction (c) that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\quad\le \alpha_n\|x_n-p\|^2+\beta_n\big\|\delta_n(z_n-p)+(1-\delta_n)\big(S^nz_n-S^np\big)\big\|^2+\lambda_n\|e_n-p\|^2\\ &\quad= \alpha_n\|x_n-p\|^2+\beta_n\big(\delta_n\|z_n-p\|^2+(1-\delta_n)\big\|S^nz_n-S^np\big\|^2\\ &\qquad -\delta_n(1-\delta_n)\big\|z_n-p-\big(S^nz_n-S^np\big)\big\|^2\big)+\lambda_n\|e_n-p\|^2\\ &\quad\le \alpha_n\|x_n-p\|^2+\beta_n\delta_n\|z_n-p\|^2+\beta_n(1-\delta_n)\big(k_n\|z_n-p\|^2\\ &\qquad +\kappa\big\|z_n-p-\big(S^nz_n-S^np\big)\big\|^2+\xi_n\big)\\ &\qquad -\beta_n\delta_n(1-\delta_n)\big\|z_n-p-\big(S^nz_n-S^np\big)\big\|^2+\lambda_n\|e_n-p\|^2\\ &\quad\le \alpha_n\|x_n-p\|^2+\beta_nk_n\|z_n-p\|^2+\xi_n\end{aligned}$$

$$
\begin{aligned}
&- \beta_n(1-\delta_n)(\delta_n-\kappa)\big\|z_n - p - \big(S^n z_n - S^n p\big)\big\|^2 + \lambda_n\|e_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + \beta_n k_n\|z_n - p\|^2 + \xi_n + \lambda_n\|e_n - p\|^2. \qquad (3.6)
\end{aligned}
$$

This implies from (3.5) that

$$
\|x_{n+1}-p\|^2 \le \|x_n-p\|^2 + (k_n-1)\|x_n-p\|^2 - \beta_n k_n \sum_{m=1}^{N} \gamma_{n,m}\|u_{n,m}-x_n\|^2 + \xi_n + \lambda_n\|e_n-p\|^2.
$$

It follows that

$$
\beta_n k_n \gamma_{n,m}\|u_{n,m} - x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (k_n - 1)\|x_n - p\|^2 + \xi_n + \lambda_n\|e_n - p\|^2.
$$

In view of the restrictions (b), (c), and (d), we obtain that

$$
\lim_{n\to\infty}\|u_{n,m} - x_n\| = 0, \quad \forall 1 \le m \le N. \qquad (3.7)
$$

Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to ω. We can get from (3.7) that $\{u_{n_i,m}\}$ converges weakly to ω for each $1 \le m \le N$. Note that

$$
F_m(u_{n,m}, u_m) + \frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \ge 0, \quad \forall u_m \in C.
$$

From the assumption (A2), we see that

$$
\frac{1}{r_{n,m}}\langle u_m - u_{n,m}, u_{n,m} - x_n\rangle \ge F_m(u_m, u_{n,m}), \quad \forall u_m \in C.
$$

Replacing n by n_i, we arrive at

$$
\frac{1}{r_{n_i,m}}\langle u_m - u_{n_i,m}, u_{n_i,m} - x_{n_i}\rangle \ge F_m(u_m, u_{n_i,m}), \quad \forall u_m \in C. \qquad (3.8)
$$

In view of (3.7) and the assumption (A4), we get that

$$
F_m(u_m, \omega) \le 0, \quad \forall u_m \in C.
$$

For any t_m with $0 < t_m \le 1$ and $u_m \in C$, let $u_{t_m} = t_m u_m + (1 - t_m)\omega$ for each $1 \le m \le N$. Since $u_m \in C$ and $\omega \in C$, we have $u_{t_m} \in C$, and hence $F_m(u_{t_m}, \omega) \le 0$ for each $1 \le m \le N$. It follows that

$$
\begin{aligned}
0 &= F_m(u_{t_m}, u_{t_m}) \\
&\le t_m F_m(u_{t_m}, u_m) + (1 - t_m)F_m(u_{t_m}, \omega) \\
&\le t_m F_m(u_{t_m}, u_m), \quad \forall 1 \le m \le N,
\end{aligned}
$$

which yields that

$$
F_m(u_{t_m}, u_m) \ge 0, \quad \forall u_m \in C.
$$

Letting $t_m \downarrow 0$ for each $1 \leq m \leq N$, we obtain from the assumption (A3) that

$$F_m(\omega, u_m) \geq 0, \quad \forall u_m \in C.$$

This implies that $\omega \in \mathrm{EP}(F_m)$ for each $1 \leq m \leq N$. This proves that $\omega \in \bigcap_{m=1}^{N} EP(F_m)$.

Next, we show that $\omega \in F(S)$. Put $\lim_{n\to\infty} \|x_n - p\| = d$, we obtain that

$$\limsup_{n\to\infty} \big\|x_n - p + \lambda_n(e_n - x_n)\big\| \leq d.$$

From (3.2), we see that

$$\limsup_{n\to\infty} \big\|y_n - p + \lambda_n(e_n - x_n)\big\| \leq d.$$

On the other hand, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_{n+1} - p\| &= \lim_{n\to\infty} \big\|(1-\beta_n)\big(x_n - p + \lambda_n(e_n - x_n)\big) + \beta_n\big(y_n - p + \lambda_n(e_n - x_n)\big)\big\| \\ &= d.\end{aligned}$$

In view of Lemma 2.3, we obtain that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.9}$$

Note that

$$\|z_n - x_n\| \leq \sum_{m=1}^{N} \gamma_{n,m}\|u_{n,m} - x_n\|.$$

This implies from (3.7) that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.10}$$

On the other hand, we have

$$\begin{aligned}&\big\|x_n - \big(\delta_n x_n + (1-\delta_n)S^n x_n\big)\big\| \\ &\quad\leq \big\|x_n - \big(\delta_n z_n + (1-\delta_n)S^n z_n\big)\big\| + \big\|\big(\delta_n z_n + (1-\delta_n)S^n z_n\big) - \big(\delta_n x_n + (1-\delta_n)S^n x_n\big)\big\| \\ &\quad\leq \|x_n - y_n\| + \delta_n\|z_n - x_n\| + (1-\delta_n)\big\|S^n z_n - S^n x_n\big\|.\end{aligned}$$

It follows from (3.9) and (3.10) that

$$\lim_{n\to\infty} \big\|x_n - \big(\delta_n x_n + (1-\delta_n)S^n x_n\big)\big\| = 0. \tag{3.11}$$

Note that

$$\begin{aligned}\big\|S^n x_n - x_n\big\| &\leq \big\|S^n x_n - \big(\delta_n x_n + (1-\delta_n)S^n x_n\big)\big\| + \big\|\big(\delta_n x_n + (1-\delta_n)S^n x_n\big) - x_n\big\| \\ &\leq \delta_n\big\|S^n x_n - x_n\big\| + \big\|\big(\delta_n x_n + (1-\delta_n)S^n x_n\big) - x_n\big\|,\end{aligned}$$

which yields that

$$(1-\delta_n)\left\|S^n x_n - x_n\right\| \leq \left\|\left(\delta_n x_n + (1-\delta_n)S^n x_n\right) - x_n\right\|.$$

This implies from the restriction (c) and (3.11) that

$$\lim_{n\to\infty}\left\|S^n x_n - x_n\right\| = 0. \tag{3.12}$$

Notice that

$$\|x_{n+1} - x_n\| \leq \beta_n\|y_n - x_n\| + \lambda_n\|e_n - x_n\|.$$

This implies from the restriction (c) and (3.9) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - S^{n+1}x_{n+1}\right\| \\ &\quad + \left\|S^{n+1}x_{n+1} - S^{n+1}x_n\right\| + \left\|S^{n+1}x_n - Sx_n\right\|.\end{aligned}$$

Since S is uniformly continuous, we obtain from (3.12) and (3.13) that

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0. \tag{3.14}$$

In view of Lemma 2.2, we obtain that $\omega \in F(S)$. This proves that $\omega \in \mathcal{F}$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ converging weakly to ω', where $\omega' \neq \omega$. In the same way, we can show that $\omega' \in \mathcal{F}$. Notice that we have proved that $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in \mathcal{F}$. Assume that $\lim_{n\to\infty}\|x_n - \omega\| = Q$, where Q is a nonnegative number. By virtue of Opial's condition of H, we have

$$Q = \liminf_{i\to\infty}\|x_{n_i} - \omega\| < \liminf_{i\to\infty}\left\|x_{n_i} - \omega'\right\| = \liminf_{j\to\infty}\left\|x_j - \omega'\right\| < \liminf_{j\to\infty}\|x_j - \omega\| = Q.$$

This is a contradiction. Hence, $\omega = \omega'$. This completes the proof. □

If $N = 1$, then Theorem 3.1 is reduced to the following.

Corollary 3.2 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $S : C \to C$ be a uniformly continuous and asymptotically κ-strict pseudocontraction in the intermediate sense. Assume that $\mathcal{F} := F(S) \cap \mathrm{EP}(F)$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{\lambda_n\}$ be sequences in $[0,1]$, and $\{e_n\}$ a bounded sequence in C. Let $\{r_n\}$ be a positive sequence such that $\liminf_{n\to\infty} r_n > 0$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in H, \\ u_n \in C \text{ such that } F(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n(\delta_n u_n + (1-\delta_n)S^n u_n) + \lambda_n e_n, \quad n \geq 1. \end{cases}$$

Assume that the following restrictions are satisfied:

(a) $\alpha_n + \beta_n + \lambda_n = 1$;

(b) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $\sum_{n=1} \xi_n < \infty$, *where* ξ_n *is defined in (2.5);*

(c) $0 < a \le \beta_n \le b < 1$, $0 \le \kappa \le \delta_n \le c < 1$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$;

where a, b, and c are some real constants. Then the sequence $\{x_n\}$ *converges weakly to some point in* $\mathcal{F}$.

If $F(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \ge 1$, then Corollary 3.2 is reduced to the following.

Corollary 3.3 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let* $S : C \to C$ *be a uniformly continuous and asymptotically* κ*-strict pseudocontraction in the intermediate sense with* $F(S) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, *and* $\{\lambda_n\}$ *be sequences in* $[0,1]$, *and* $\{e_n\}$ *a bounded sequence in C. Let* $\{r_n\}$ *be a positive sequence such that* $\liminf_{n\to\infty} r_n > 0$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + \beta_n(\delta_n P_C x_n + (1 - \delta_n) S^n P_C x_n) + \lambda_n e_n, \quad n \ge 1. \end{cases}$$

Assume that the following restrictions are satisfied:

(a) $\alpha_n + \beta_n + \lambda_n = 1$;

(b) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $\sum_{n=1} \xi_n < \infty$, *where* ξ_n *is defined in (2.5);*

(c) $0 < a \le \beta_n \le b < 1$, $0 \le \kappa \le \delta_n \le c < 1$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$;

where a, b, and c are some real constants. Then the sequence $\{x_n\}$ *converges weakly to some point in* $F(S)$.

For the class of asymptotically nonexpansive mappings in the intermediate sense, we can obtain from Theorem 3.1 the following results immediately.

Corollary 3.4 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let* F_m *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies (A1)-(A4) for each* $1 \le m \le N$, *where* $N \ge 1$ *is some positive integer. Let* $S : C \to C$ *be a uniformly continuous and asymptotically nonexpansive mapping in the intermediate sense. Assume that* $\mathcal{F} := F(S) \cap \bigcap_{m=1}^{N} EP(F_m)$ *is nonempty. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, *and* $\{\lambda_n\}$ *be sequences in* $[0,1]$, *and* $\{e_n\}$ *a bounded sequence in C. Let* $\{r_{n,m}\}$ *be a positive sequence such that* $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $\{\gamma_{n,m}\}$ *a sequence in* $[0,1]$ *for each* $1 \le m \le N$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in H, \\ u_{n,m} \in C \text{ such that } F_m(u_{n,m}, u_m) + \frac{1}{r_{n,m}} \langle u_m - u_{n,m}, u_{n,m} - x_n \rangle \ge 0, \quad \forall u_m \in C, \\ z_n = \sum_{m=1}^{N} \gamma_{n,m} u_{n,m}, \\ x_{n+1} = \alpha_n x_n + \beta_n(\delta_n z_n + (1 - \delta_n) S^n z_n) + \lambda_n e_n, \quad n \ge 1. \end{cases}$$

Assume that the following restrictions are satisfied:

(a) $\alpha_n + \beta_n + \lambda_n = 1$;

(b) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ *and* $\sum_{n=1} \xi_n < \infty$, *where*

$$\xi_n = \max\Big\{0, \sup_{x,y\in C}\big(\|S^n x - S^n y\|^2 - k_n\|x-y\|^2\big)\Big\};$$

(c) $0 < a \le \beta_n \le b < 1$, $0 \le \kappa \le \delta_n \le c < 1$ *and* $\sum_{n=1}^{\infty}\lambda_n < \infty$;

(d) $\sum_{m=1}^{N}\gamma_{n,m} = 1$ *and* $0 < d \le \gamma_{n,m} \le 1$ *for each* $1 \le m \le N$,

where a, b, c, and d are some real constants. Then the sequence $\{x_n\}$ *converges weakly to some point in* $\mathcal{F}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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References

1. Chang, SS, Yao, JC, Kim, JK, Yang, L: Iterative approximation to convex feasibility problems in Banach space. Fixed Point Theory Appl. **2007**, Article ID 046797 (2007)
2. Qin, X, Cho, SY, Kang, SM: Common fixed points of a pair of non-expansive mappings with applications to convex feasibility problems. Glasg. Math. J. **52**, 241-252 (2010)
3. Ye, J, Huang, J: Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces. J. Math. Comput. Sci. **1**, 1-18 (2011)
4. Cho, SY, Kang, SM: Approximation of fixed points of pseudocontraction semigroups based on a viscosity iterative process. Appl. Math. Lett. **24**, 224-228 (2011)
5. Kim, JK, Cho, SY, Qin, X: Hybrid projection algorithms for generalized equilibrium problems and strictly pseudocontractive mappings. J. Inequal. Appl. **2010**, Article ID 312602 (2010)
6. Qin, X, Chang, SS, Cho, YJ: Iterative methods for generalized equilibrium problems and fixed point problems with applications. Nonlinear Anal., Real World Appl. **11**, 2963-2972 (2010)
7. Kim, JK, Nam, YM, Sim, JY: Convergence theorems of implicit iterative sequences for a finite family of asymptotically quasi-nonexpansive type mappings. Nonlinear Anal. **71**, e2839-e2848 (2009)
8. Lau, ATM, Miyake, H, Takahashi, W: Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces. Nonlinear Anal. **67**, 1211-1225 (2007)
9. Lau, AT, Takahashi, W: Invariant submeans and semigroups of nonexpansive mappings on Banach spaces with normal structure. J. Funct. Anal. **142**, 79-88 (1996)
10. Shioji, N, Takahashi, W: Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces. Proc. Am. Math. Soc. **125**, 3641-3645 (1997)
11. Bruck, RE, Kuczumow, T, Reich, S: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**, 169-179 (1993)
12. Kirk, WA: Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type. Isr. J. Math. **17**, 339-346 (1974)
13. Schu, J: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991)
14. Sahu, DR, Xu, HK, Yao, JC: Asymptotically strict pseudocontractive mappings in the intermediate sense. Nonlinear Anal. **70**, 3502-3511 (2009)
15. Qin, X, Cho, YJ, Kang, SM: Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J. Comput. Appl. Math. **225**, 20-30 (2009)
16. Lv, S: Generalized systems of variational inclusions involving (A, η)-monotone mappings. Adv. Fixed Point Theory **1**, 1-14 (2011)
17. Qin, X, Cho, SY, Kang, SM: Strong convergence of shrinking projection methods for quasi-ϕ-nonexpansive mappings and equilibrium problems. J. Comput. Appl. Math. **234**, 750-760 (2010)
18. Kim, JK, Kim, KS: A new system of generalized nonlinear mixed quasivariational inequalities and iterative algorithms in Hilbert spaces. J. Korean Math. Soc. **44**, 823-834 (2007)
19. Kim, JK: Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings. Fixed Point Theory Appl. **2011**, 10 (2011)

20. Kim, JK, Cho, SY, Qin, X: Some results on generalized equilibrium problems involving strictly pseudocontractive mappings. Acta Math. Sci. **31**, 2041-2057 (2011)
21. Yang, S, Li, W: Iterative solutions of a system of equilibrium problems in Hilbert spaces. Adv. Fixed Point Theory **1**, 15-26 (2011)
22. Qin, X, Cho, SY, Kang, SM: On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings. Appl. Math. Comput. **215**, 3874-3883 (2010)
23. Takahashi, W, Toyoda, M: Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **118**, 417-428 (2003)
24. Tada, A, Takahashi, W: Weak and strong convergence theorems for a nonexpansive mappings and an equilibrium problem. J. Optim. Theory Appl. **133**, 359-370 (2007)
25. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123-145 (1994)
26. Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117-136 (2005)
27. Schu, J: Weak and strong convergence of fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc. **43**, 153-159 (1991)
28. Tan, KK, Xu, HK: Approximating fixed points of nonexpansive mappings by the Ishikawa iterative process. J. Math. Anal. Appl. **178**, 301-308 (1993)
29. Hao, Y, Cho, SY, Qin, X: Some weak convergence theorems for a family of asymptotically nonexpansive nonself mappings. Fixed Point Theory Appl. **2010**, Article ID 218573 (2010)

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